Other examples of Group actions.

① A group G acts on itself by

ⓐ left multiplication:

For $g \in G$, $x \in G$

$$g \cdot x := gx$$

Clearly $e \cdot x = ex = x$ ✓ if e is the identity.

$\forall g_1, g_2 \in G$, $x \in G$.

$$g_1 \cdot (g_2 \cdot x) = g_1 \cdot (g_2 x) = g_1(g_2 x)$$
$$= (g_1 g_2) x = (g_1 g_2) \cdot x. \checkmark$$

ⓑ right multiplication by inverse:

For $g \in G$, $x \in G$

We define $g \cdot x = x g^{-1}$.

We check (1) $e \cdot x = x e^{-1} = x e = x$. ✓ if e is the identity of G.

(2) $\forall g_1, g_2 \in G$, $x \in G$,

$$g_1 \cdot (g_2 \cdot x) = g_1 \cdot (x g_2^{-1}) = (x g_2^{-1}) g_1^{-1}$$
$$= x (g_2^{-1} g_1^{-1}) = x (g_1 g_2)^{-1}$$
$$= (g_1 g_2) \cdot x. \checkmark$$

Notice that $\forall x \in G$, the orbit by left mult is $Gx = \{g \cdot x : g \in G\} = G$.

<u>Proof:</u> Clearly $Gx \subseteq G$. Given any $y \in G$, If we let $g = y x^{-1}$, then

$$g \cdot x = y x^{-1} x = y. \quad \therefore Gx = G. \;\square$$

The isotropy: For $x \in G$, the isotropy $G_x = \{g \in G : g \cdot x = x\}$.

Note $g \cdot x = gx = x$
$\Rightarrow gxx^{-1} = xx^{-1} \Rightarrow ge = e$ (where e is the identity in G.)
$\Rightarrow g = e$.
$\therefore G_x = \{e\}$

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② Another way that a group G can act on itself is by conjugation.

$\forall g \in G$, $x \in G$, we define

$g \cdot x = gxg^{-1}$.

the orbit $Gx = \{gxg^{-1} : g \in G\}$ is called the <u>conjugacy class</u> $cl(x)$ of x.

The stabilizer of x is $G_x = \{g \in G : g \cdot x = gxg^{-1} = x\}$
$= \{g \in G : gx = xg\}$
$= C_G(x) =$ <u>centralizer of x in G</u>

<u>Example:</u> $G = S_4$, $x = (1,2)$

Let's calculate Gx and G_x if G acts by conjugation.

$$Gx = \{\sigma(1,2)\sigma^{-1} : \sigma \in S_4\}$$

e.g. $(3,4)(1,2)(3,4)^{-1} = (3,4)(1,2)(3,4) = (3,4)(3,4)(1,2) = (1,2)$.

disjoint cycles commute

$(1,4)(1,2)(1,4)^{-1} = (1,4)(1,2)(1,4) = (2,4) = (4,2)$

$(1,3,2,4)(1,2)(1,3,2,4)^{-1} = (1,3,2,4)(1,2)(4,2,3,1)$
$= (3,4)$

③ ② ①
$\sigma\ (1,2)\ \sigma^{-1}$ plug

$\sigma(1) \overset{①}{\longmapsto} 1 \overset{②}{\to} 2 \overset{③}{\to} \sigma(2)$ $\quad \sigma(3) \to 3 \to 3 \to \sigma(3)$

$\sigma(2) \to 2 \to 1 \to \sigma(1)$ $\quad \sigma(4) \to 4 \to 4 \to \sigma(4)$

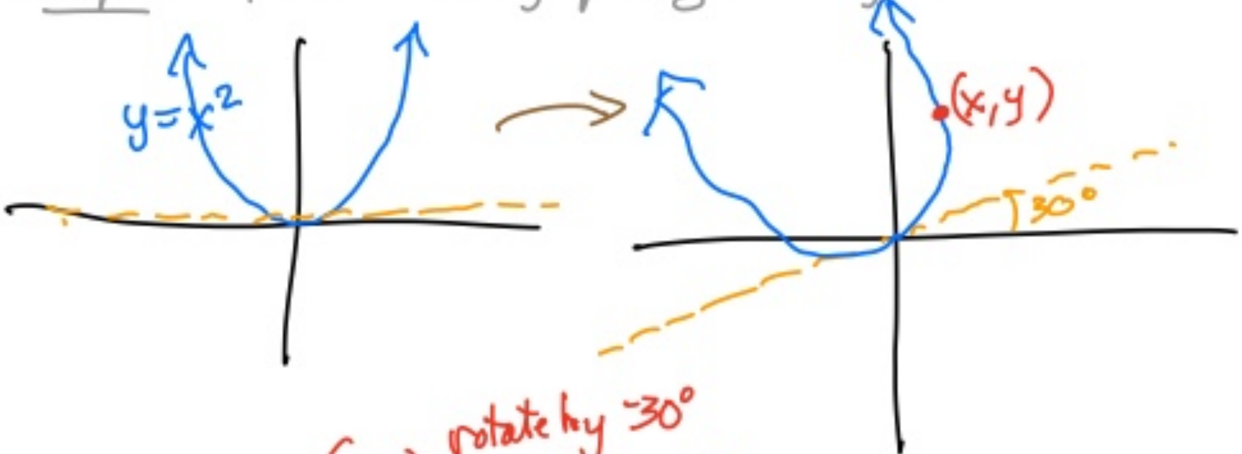
$\Rightarrow \quad \sigma(1,2)\sigma^{-1} = (\sigma(1), \sigma(2))$

In general, for any permutation τ

$\sigma\tau\sigma^{-1} = \tau$ permutation with
1 replaced by $\sigma(1)$,
2 replaced by $\sigma(2)$,
3 replaced by $\sigma(3)$,
4 replaced by $\sigma(4)$

Similarly if f & g are functions from $\mathbb{R}^2 \to \mathbb{R}^2$, 1-1, onto.
then $x \mapsto g(f(g^{-1}(x)))$ is the function f with the $g(x)$ variables plugged.

Example: Rotate the graph $y = x^2$ by $30°$.



$(x,y) \xrightarrow{\text{rotate by } -30°}$ then it should satisfy $y = x^2$.

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos(30°) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$g^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos -30° & \\ \vdots & \ddots \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{\sqrt{3}}{2}x + \frac{1}{2}y \\ -\frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{pmatrix}$$

New equation $\tilde{x}^2 = \tilde{y}$

$$\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right) = \left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)^2.$$

Back to our question: what is the conjugation orbit of $(1,2)$ in S_4?

$$G(1,2) = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

all possible transpositions.

$|G(1,2)| = 6$

$$G_{(1,2)} = \{\sigma \in S_n : \sigma (1,2) \sigma^{-1} = (1,2)\}$$

$$= \{e, (3,4), (1,2), (1,2)(3,4)\}$$

$|G_{(1,2)}| = 4$

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example

$$G(1,2,3) = \{(1,2,3), (1,2,4), (1,3,4), (2,3,4), (1,3,2), (1,4,3), (1,4,2), (2,4,3)\}.$$

all possible 3 cycles

$|G(1,2,3)| = 8$

$$G_{(1,2,3)} = \{\sigma \in S_4 : \sigma (1,2,3) \sigma^{-1} = (1,2,3)\}.$$

$$= \{e, (1,2,3), (3,2,1)\}$$

$|G_{(1,2,3)}| = 3$

<u>Orbit Stabilizer Theorem</u> -

Let the group G act on the set X. Let $x \in X$. Then the orbit Gx is in 1-1 correspondence with $\underbrace{G/G_x}_{\text{set of cosets}}$

in particular, if G is finite,

$$\frac{|G|}{|G_x|} = |Gx|$$